XV The slope formula

$K = \bar{K}$, $X/\mathrm{Spec}\,K$ smooth curve, $\hat{X}$ sm proj closure, $D := \hat{X} \setminus X$

Sst vs for X: $V \subseteq X^{an}$ fin set of (2) pts, $X^{an} \setminus V = \coprod \mathring{\mathbb{D}}^{an} \amalg \coprod \mathring{\mathbb{D}}^{an} \setminus \{0\} \amalg \coprod_{i=1}^{r} \mathring{A}(a_i)^{an}$

$\{\text{sst vs } V \text{ for } \hat{X}\} \underset{bij}{\cong} \{\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}_K \text{ sst models}\}$

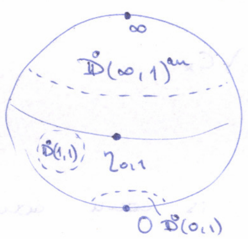
$(sp^{an})^{-1}(\text{gen pt of } |\mathfrak{X}|) \longleftrightarrow \mathfrak{X}$

Graph of V: $\Sigma(X,V) = V \amalg \coprod_{D} \underbrace{\Sigma(\mathring{\mathbb{D}}^{an} \setminus \{0\})}_{\cong (0,\infty)} \amalg \coprod_{i=1}^{r} \underbrace{\Sigma(\mathring{A}(a_i)^{an})}_{\cong (0, v(a_i))}$

Ex. 1) If $\hat{X}$ has good reduction (i.e. sm proper $\mathcal{X}/\mathrm{Spec}\,\mathcal{O}_K$)

$\Rightarrow (sp^{an})^{-1}(\text{gen pt of } \mathcal{X}_k) = \{pt\}$ is sst vs

Eg. $\mathbb{P}^1$



$\Big\} \; \mathbb{D}(0,1) \xrightarrow{sp} \mathbb{A}^1_k$ $\quad x$: img of closed pt of $\mathrm{Spa}(L, L^+) \to \mathbb{P}^{1,ad}$

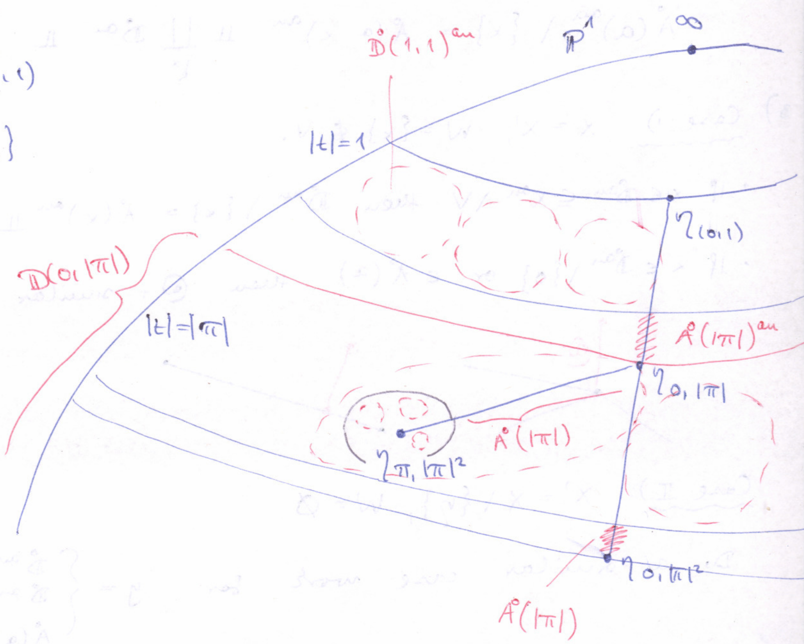
$sp(x)$ = img of closed pt of induced $\mathrm{Spa}\,L^+ \to \mathbb{P}^1_{\mathcal{O}_K}$

We get that for $\mathbb{P}^1$: $sp^{-1}(\bar{x} \in \mathbb{P}^1_k(k)) = \overline{\mathring{\mathbb{D}}(x,1)}$ closure, adds a rk 2 pt

$sp^{-1}(\eta) = \{\eta_{0,1}\}$

For $\mathbb{P}^{1\,an}$ we get: $(sp^{an})^{-1}(\bar{x}) = \mathring{\mathbb{D}}(x,1)$

$(sp^{an})^{-1}(\eta) = \{\eta_{0,1}\}$



§1 Skeletal points on X^{an}

Prop. V sst vs for X, $\emptyset \neq X' \subseteq X$ open, $V' \subseteq V$

1) if V' sst vs for X' and $V \subseteq V'$ then $\Sigma(X,V) \subseteq \Sigma(X',V')$

and $\tau_{\Sigma(X,V)} \circ \tau_{\Sigma(X',V')} = \tau_{\Sigma(X,V)}$

2) if $V' \subseteq \Sigma(X,V)$ fin set of type (2) pts then $V \cup V'$ is a sst vs and $\Sigma(X, V\cup V')$ is a refinement of $\Sigma(X,V)$

3) $W \subseteq X^{an}$ fin set of type (2) pts then $\exists V'$ sst vs for X', $W \cup V \subseteq V'$

Pf: 1) Lemma^{PO} (BPR Lemma 3.14) $\Sigma(X,V) = \{x \in X^{an} \mid \nexists\, x \in \mathbb{D}^{an} \subseteq X \setminus V \text{ open nbh}\}$

The claim $\Sigma(X,V) \subseteq \Sigma(X',V')$ is immediate from this Lemma.

Recall the def of τ: for $x \in X^{an} \setminus \Sigma(X,V)$, $\tau_{\Sigma(X,V)}(x) = \partial C_x$ where C_x is the conn component of x in $X^{an} \setminus \Sigma(X,V)$ and $\{\partial C_x\} = \overline{C_x} \setminus C_x$

Then the claim about τ follows from def:

$x \in (X')^{an} \setminus \Sigma(X',V') \supseteq C'_x$ conn comp of x, then $C'_x \subseteq C_x$ is conn comp of $x \in \Sigma(X,V) \Rightarrow C_x = C_{\partial C'_x}$

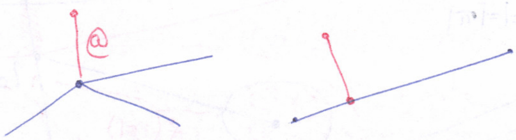
$\mathring{A}(a)^{an}$

2) Wlog $V' = \{x\}$ s.t. x is on an edge of $\Sigma(X,V)$. Then the claim follows from explicit calculation:

$$\mathring{A}(a)^{an} \setminus \{x\} = \mathring{A}(1,x)^{an} \sqcup \coprod_{k^\times} \mathring{\mathbb{D}}^{an} \sqcup \mathring{A}(x,a)^{an} \qquad @$$

3) Case I): $X = X'$, $W = \{x\} \not\subseteq V$.

- If $x \in \mathring{\mathbb{D}}^{an} \subseteq X^{an} \setminus V$ then $\mathring{\mathbb{D}}^{an} \setminus \{x\} = \mathring{A}(x)^{an} \sqcup \coprod_{k^\times} \mathring{\mathbb{D}}^{an} \Rightarrow V \cup \{x\}$ sst vs
- If $x \in \mathring{\mathbb{D}}^{an} \setminus \{0\}$ or $\in \mathring{A}(a)$ then @ + similar implies $V \cup \{x, \tau_\Sigma(x)\}$



Case II): $X' = X \setminus \{y\}$, $W = \emptyset$

Do a similar case work for $y = \begin{cases} \mathring{\mathbb{D}}^{an} \\ \mathring{\mathbb{D}}^{an} \setminus \{0\} \\ \mathring{A}(a)^{an} \end{cases}$ □

Def. Skeletal pts of X^{an}: $H(X^{an}) := \{\text{type (2) (3) pts of } X^{an}\}$

Cor. $H(X^{an}) = \varinjlim_V \Sigma(X,V)$.

For type (2) pts, this follows directly from Prop.

For type (3) pts, one explicitly writes down suitable type (2) pts in $\begin{cases} \mathring{\mathbb{D}}^{an} \\ \mathring{\mathbb{D}}^{an} \setminus \{0\} \\ \mathring{A}(a)^{an} \end{cases}$

§2 The genus formula

Recall: $g(\hat{X}) = h^1(\hat{X}, \mathcal{O}_{\hat{X}}) = \dim H^1(\hat{X}, \mathcal{O}_{\hat{X}})$ genus of a smooth proj curve

Def. • If $x \in X^{an}$ type (2) pt, $\{x\} = (sp^{an})^{-1}(\eta_{C_x})$, $C_x \subseteq |\mathfrak{X}|$ an curve for one suitable model then $g(x) := g(C_x) = g(\widetilde{\mathcal{H}(x)})$ where $\mathcal{H}$ is the function field over k.

• For a graph Σ, $g(\Sigma) = \mathrm{rk}_{\mathbb{Z}} H_1(\Sigma, \mathbb{Z})$

Thm. (Bosch – Lütkebohmert) If V sst vs for $\hat{X}$ then $g(\hat{X}) = \sum_{x \in V} g(x) + g(\Sigma(X,V))$

§3 Metric structure on $H(X^{an})$

Recall: $\Sigma(X,V)$ has metric structure via $\Sigma(\mathring{A}(a)^{an}) \cong (0, v(a))$

Pf of Prop from §1 $\Rightarrow$ if $V \subseteq V'$ sst vs then $\Sigma(X,V) \hookrightarrow \Sigma(X,V')$ is isometric, i.e. each edge isometric into edge.

Def. Path: $[a,b] \xrightarrow{p} \Sigma(X,V)$ continuous s.t. $p^{-1}V \subseteq [a,b]$ is discrete and $p|_{[a,b]\setminus p^{-1}V}$ is locally an isometry.

Def. Path distance metric on $\Sigma(X,V)$: $d(x,y) := \min\limits_{\substack{[a,b] \to \Sigma(X,V) \\ \text{path},\ a \mapsto x,\ b \mapsto y}} \{b-a\}$

Lemma. $[x,y] \subseteq \Sigma(X,V)$ is a shortest path from x to y and $V' \subseteq V$ then $[x,y]$ is also a shortest path in $\Sigma(X,V')$.

Pf: {Assume $[x,y]' \subseteq \Sigma(X,V')$ is shorter. Then $[x,y]' \circ [x,y] \in H_1(\Sigma(X,V), \mathbb{Z})$ does not come from $H_1(\Sigma(X,V), \mathbb{Z})$, but this is impossible by the genus formula. }

Cor. Get natural metric on $H(X^{an})$.

Rmk. 1) If $x,y \in H(X^{an})$ not in the same conn component of $X^{an} \setminus \Sigma(X,V)$, then any path from x to y passes through $\Sigma(X,V)$.

$\Rightarrow d(x,y) = d(x, \tau(x)) + d(\tau(x), \tau(y)) + d(\tau(y), y)$.

2) Exc. On $H(\mathbb{A}^{1,an})$ the distance is as follows:

$$d(\eta_{a_1,r_1}, \eta_{a_2,r_2}) = \begin{cases} \log \dfrac{\max(r_1,r_2)}{\min(r_1,r_2)} \\ 2v(a_1-a_2) + \log r_1 + \log r_2 \end{cases}$$

3) The path distance metric on annuli and disks is canonical, i.e. preserved under any automorphism.

§4 Slope formula. ← analogue of the balancedness condition

<u>Def.</u> Geodesic on $H(X^{an})$: isometric embedding $[a,b] \xhookrightarrow{\rho} H(X^{an})$

Tangent space at $x \in H(X^{an})$: $T_x := \left\{ [a,b] \xhookrightarrow{\rho} H(X^{an}) \,\middle|\, \text{geodesic}, \ \rho(0) = x \right\} / \sim$

where $\rho \sim \rho' \iff \exists \varepsilon'' > 0 : \rho|_{[0,\varepsilon'']} = \rho'|_{[0,\varepsilon'']}$

<u>Lemma.</u> 1) x type (3) $\Rightarrow |T_x| = 2$.

2) x type (2) $\Rightarrow T_x \cong \mathcal{C}_x(k)$.

Pf: See BPR Lemma 5.12 for a full proof.

Correspondence in 2): $\exists\ \mathfrak{X}$ strongly sst model s.t. $x \in V(\mathfrak{X})$.

* see erratum on p. 67 — need this in order to avoid self-intersections

- Given $\tilde{x} \in \mathcal{C}_x(k)$, after blowing up (wma $\tilde{x}$ is a double pt)

 Then for $y := (sp^{an})^{-1}$ (gen pt of the 2nd curve through $\tilde{x}$), $[x,y]$ represents the tangent direction

- Conversely given $[\rho: [0,\varepsilon] \longrightarrow H(X^{an})] \in T_x$ choose type (2) pt $y \in \mathrm{Im}(\rho) \setminus \{x\}$,

 choose a model where both x and y come from generic pts.

 $\exists!$ shortest way to pass from $\mathcal{C}_x$ to $\mathcal{C}_y$ when $\varepsilon < \min\{\ell(L) \mid L \subseteq H(X^{an})$ nontrivial loop$\}$.

 Then $\tilde{x} :=$ first intersection point.

<u>Def.</u> 1) $F: X^{an} \longrightarrow \mathbb{R}$ is pw linear if $\forall \rho: [a,b] \hookrightarrow H(X^{an})$ geodesic: $F \circ \rho$ is pw lin.

For $v \in T_x$, $d_v F(x) := \lim_{\varepsilon \to 0} (F \circ \rho)'(\varepsilon)$ when F is pw lin.

2) $F: X^{an} \to \mathbb{R}$ pw lin balanced in $x \in H(X^{an})$ (or harmonic) if

$d_v F(x) \neq 0$ for fin many $v \in T_x$ only, and $\sum_{v \in T_x} d_v F(x) = 0$.

<u>Thm.</u> (Slope formula) $f \in \mathcal{O}_X(X)^\times$, e.g. X is affine, $F := -\log|f| : X^{an} \to \mathbb{R}$,

$V :=$ sst vs for X. Then

1) F factors via $\tau_{\Sigma(X,V)}$

2) F is pw lin and linear on edges of $\Sigma(X,V)$ with integer slopes

3) if x is type (2) then $d_v F(x) = \mathrm{ord}_v(\tilde{f}_x)$ where $\tilde{f}_x \in \widetilde{\mathcal{H}(x)}$ img of $cf \in \mathcal{H}(x)^+$ with $c \in K^\times$, $|c| = |f(x)|^{-1}$, $\tilde{f} :=$, so $\tilde{f} \in \mathcal{O}^+_{X,x}$

4) F is balanced

5) for $x \in D$, $e \subseteq \Sigma(X,V)$ ray leading to x, $y \in V$ vertex of e $\to d_e F(y) = \mathrm{ord}_x(f)$.

Pf: 1): f unit $\Rightarrow \begin{cases} f \text{ unit on } \mathbb{D}^{an} \Rightarrow |f| \text{ is constant} \\ f \text{ unit on } \mathring{A}(a) \text{ or } \mathring{\mathbb{D}} \setminus \{0\} \Rightarrow |f| \text{ factors via sk map } \sigma \end{cases}$

(proven in XIII)

$\Rightarrow$ F factors via $\tau_{\Sigma(X,V)}$ ✓

2): as in 1) ✓

3):

XVI Poincaré – Lelong 07.12.2018

<u>Erratum</u>*: Given $[[0,\varepsilon] \xrightarrow{p} H(X^{an})] \in T_x$:

- Shrink ε s.t. $\varepsilon < \frac{1}{2} \min \{ \text{len}(L) \mid L \text{ loc geod loop not null-homologous} \}$
- Pick y type (2) pt $\in \operatorname{Im} p \setminus \{x\}$
- as last time

- Refining V, we may assume that v is given by edge $[x,y] \subseteq \Sigma$, $\cong [0, v(a)]$

F is linear on $[x,y]$ of integer slope $=: n$ by 2)

$\Rightarrow F(y) = n \cdot v(a) + F(x)$, $\quad F(x) = -v(c)$

WMA $n \geq 0$ by replacing f by f^{-1} if necessary.

- Let $\mathfrak{X} :=$ model associated to V, $\tilde{x} := C_x \cap C_y$, and fix $\hat{\mathcal{O}}_{\mathfrak{X}, \tilde{x}} \cong \mathcal{O}_K [\![X,Y]\!] / (XY - a)$ where C_x is defined by $Y = 0$.

- <u>Claim</u>: $\tilde{f}$ is given by some elt of $\mathcal{O}_K[\![X,Y]\!]/(XY-a)$ and $\tilde{f} \equiv^{(\text{unit})} X^n \mod (m_K, Y)$

(This will settle 3).)

Pf: $|\tilde{f}| = |X^n|$ on $\mathring{A}(a)^{an}$

Hence $u := \tilde{f} X^{-n}$ has $|u| = 1$ on $\mathring{A}(a)^{an}$.

Write $u = \dots + c_{-2} Y^{+2} + c_{-1} Y^{+1} + c_0 + c_1 X + c_2 X^2 + \dots$ (we used $XY = a$ to write the negative terms with Y)

If $r \in \Sigma(\mathring{A}(a)^{an})$, $|u(r)| = \max\limits_{i,j \geq 0} \left\{ |c_i| r^i, |c_{-j}| \left(\frac{|a|}{r} \right)^j \right\} = 1 \quad \forall r$

$\Rightarrow c_i, c_{-i} \in \mathcal{O}_K \quad \forall i$

Thus $\tilde{f} = u X^n \in \mathcal{O}_K[\![X,Y]\!] / _{XY-a}$

$u \equiv c_0 + c_1 X + \dots \mod Y$ with $\underbrace{c_0 \in \mathcal{O}_K^\times}$

this reflects that $\tilde{f}$ is not uniquely determined, but up to unit only

In particular: u is a unit mod (Y, m_K)

$\Rightarrow \tilde{f} \equiv (\text{unit}) \cdot X^n \mod (Y, m_K) \Rightarrow \operatorname{ord}_v(\tilde{f}_x) = n$ as claimed in 3)

Rmk. The pf shows that a unit on $A(a)$ or $A^\circ(a)$ has the form

$c \cdot X^n \cdot (1+\varepsilon)$ where $c \in K^\times$, $n \in \mathbb{Z}$ and $|\varepsilon| < 1$.

4): Immediate from 3): we have seen that

$$\sum_{v \in C_x(k)} d_v F(x)[v] = \operatorname{div}(\tilde{f}_x)$$

is principal, hence has degree 0.

5): Similar to 3) but with punctured disk instead of annulus.

$\hat{\mathfrak{X}} :=$ model of $\hat{X}$ def'd by V

$\tilde{x} :=$ specialisation of x, $\tilde{x} \in \hat{\mathfrak{X}}(k)$ } this part is superfluous

Smooth point since $(sp^{an})^{-1}(\tilde{x}) = \mathring{\mathbb{D}}^{an}$

Wma $n := \operatorname{ord}_x(f) \geq 0$

$\Rightarrow$ f extends over $X \cup \{x\}$ and is of the form

$c \cdot X^n \cdot u$ for $c \in K^\times$, $|u| = 1$.

$\Rightarrow$ slope of $F|_e$ is n $\square$

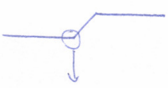
§1 Comparison with the tropical definition

Recall: $F : X^{an} \longrightarrow \mathbb{R}$ is piecewise smooth if $\forall x \in X^{an}$ $\exists x \in (U, \varphi, V)$ tropical chart and $\psi : \mathbb{R}^r \longrightarrow \mathbb{R}$ s.t. $F|_V = \psi \circ \operatorname{trop}_\varphi|_V$ (cf. p. 50)

Prop. A smooth function F on X^{an} is piecewise balanced / harmonic in the following sense:

- $\forall p : [0, b] \longrightarrow H(X^{an})$ geodesic: $F \circ p$ is smooth with Γ-rational break pts
- $\forall x \in H(X^{an})$: $\sum_v d_v F(x) = 0$ (in ptic, the sum is finite)

Even more: $\sum_v d_v F(x)[v] \in \operatorname{Pic}(C_x)_{\mathbb{R}}$ is trivial.

Cor. The function  on $\mathbb{D}$ from VII is not smooth. Because

this type (2) point is not balanced. But it is pw smooth $\square$

PF OF PROP: From defs.

Given $x \in \operatorname{Im}(p)$, find trop chart $x \in (U, \varphi, W)$ as in the defn of pw smooth.

$\varphi = (\varphi_1, \dots, \varphi_r) \in \mathcal{O}_U(U)^\times$, $V :=$ sst vs for U s.t. $\Sigma \supseteq \operatorname{Im} p$

By Thm. 1): $\operatorname{trop}_\varphi$ factors via τ_Σ, $\Sigma := \Sigma(U, V)$

By 2): $F \circ p$ is pw smooth

Balancedness follows from 3).
for φ_i

If F is in a chart given by ψ, get $d_v F(x) = -\sum_i \left(\frac{\partial}{\partial x_i} \psi\right)(\operatorname{trop}_{\varphi_i} x) \cdot d_v \log|\varphi_i|$

$$\sum_v \text{---} = \sum_i (\partial_i \psi)(\operatorname{trop}_{\varphi_i} x) \sum_v d_v \log|\varphi_i| = 0$$

Partial converse. A pw lin F on X^{an} is locally of the form $-\log|f|\cdot\mathbb{R}$ for unit f $\Leftrightarrow$ $\forall x\in H(X^{an})$ balanced & x of type (2): $\underbrace{\sum_v d_v F(x)[v] \in \operatorname{Pic}(\mathcal{C}_x)_{\mathbb{Q}} \text{ is trivial.}}$

automatic if all $\mathcal{C}_x \cong \mathbb{P}^1$ or if $k=\mathbb{F}$

Prop. $F: X^{an}\to\mathbb{R}$ is pw smooth in the tropical sense $\Leftrightarrow$
$\Leftrightarrow$ locally τ_Σ composed w/ pw smooth on Σ w/ break pts of type (2).

Rmk. Trying to prove a full converse leads to the following question.
If $\Sigma\subseteq\mathbb{R}^r$ is some graph, $\psi:\Sigma\to\mathbb{R}^r$ cont & smooth on edges, when $\exists\tilde\psi$ smooth on $\mathbb{R}^r$ s.t. $\tilde\psi|_\Sigma=\psi$? (M. doesn't know how to do this.)

§2 Currents

$X/\operatorname{Spec}K$ loft pure d-dim sep

Recall: $\mathcal{A}^{p,q}(W)$, $W\subseteq X^{an}$ open, ω given by covers $W=\bigcup_i (U_i,\varphi_i,V_i)$ and $\omega_i\in\mathcal{A}^{p,q}(\operatorname{trop}\varphi_i(V_i))$ and gluing conditions.

Recall: $\mathcal{A}^{p,q}_c(W)\xrightarrow{\int}\mathbb{R}$. Any $\omega\in\mathcal{A}^{p,q}_c(W)$ is rep'd by a single trop chart $(U,\varphi,V,\tilde\omega)$

$$\int_X\omega=\int_{\operatorname{trop}_\varphi U}\tilde\omega$$

$\operatorname{trop}_\varphi U$ ⟵ as trop cycle

Def. Topology on $\mathcal{A}^{p,q}_c(W)$: a net $\omega_k\to\omega$ converges iff $\exists$ fin many trop charts $(U_i,\varphi_i,V_i)\subseteq W$ in which all the ω_k and ω can be expressed, represented by $\omega_{k,i}$ and ω_i, and fin many $\Delta_i\subseteq\mathbb{R}^n$ compact polyhedra s.t. $\operatorname{supp}\omega_{k,i}$, $\operatorname{supp}\omega_i\subseteq\Delta_i$ $(\forall k)$ and $\omega_{k,i}\to\omega_i$ in the sense we had before. (on all d-dim polyhedra in $\operatorname{trop}_{\varphi_i}(U_i)\cap\Delta_i$)

Def. $D_{p,q}(W):=$ top dual of $\mathcal{A}^{p,q}_c(W)=\operatorname{Hom}_{Top}(\mathcal{A}^{p,q}_c(W),\mathbb{R})$ currents

For $W'\subseteq W$, we get $D_{p,q}(W')\longrightarrow D_{p,q}(W)$ dual to $\mathcal{A}^{p,q}_c(W)\hookrightarrow\mathcal{A}^{p,q}_c(W')$.

Def. $D^{p,q}:=D_{d-p,d-q}$ This comes with d,d',d'' obtained by dualisation of the corresponding operators on $\mathcal{A}^{p,q}_c$. | This makes $D_{p,q}$ a shea

Ex. 1) $A^{p,q}(W)\overset{@}{\hookrightarrow}D^{p,q}(W)$

$\omega\longmapsto[\omega]$ where $[\omega](\eta)=\int_X\omega\wedge\eta$

$$(d'[\omega])(\eta)=\int_X\omega\wedge d'\eta=(-1)^{p+q}\Big(\underbrace{\int_X d'(\omega\wedge\eta)}_{=0\text{ by Stokes}}-\int_X d'\omega\wedge\eta\Big)=\Big[(-1)^{p+q}d'\omega\Big](\eta)$$

$\Rightarrow$ @ commutes w/ d,d',d'' up to $(-1)^{p+}$

2) $i: Y \hookrightarrow X$ closed subvty, purely s-dim

$\delta_Y \in D^{d-s,d-s}(X)$, $\delta_Y(\eta) := \int_Y i^*\eta$

Special cases: $\delta_X = \int_X \in D^{0,0}(X)$ whole space

$\delta_x \in D^{d,d}(X)$ $x \in X(k)$ single point, $\delta_x(f) = f(x)$ (Dirac)

3) $A^{r,s} \otimes D^{p,q} \longrightarrow D^{r+p,s+q}$

$\omega \otimes T \longmapsto \omega \wedge T := [\eta \longmapsto T(\omega\wedge\eta)]$

§3 Poincaré – Lelong for curves

$X/\operatorname{Spec} K$ conn smooth curve

Lemma. $\omega \in A^{1,1}_c(X)$ rep by $(U, \varphi, V, \tilde{\omega})$, V sst σ for U

View $e \in \Sigma$ as a Γ-rat polyhedron via $\mathring{A}(a) \longrightarrow \mathbb{G}_m$

Then $\operatorname{trop}\varphi|_e^*\tilde{\omega}$ is a $(1,1)$-form on e and $\int_X \omega = \int_\Sigma \operatorname{trop}\varphi|_\Sigma^*\omega := \sum_{e\in\Sigma} \operatorname{trop}\varphi|_e^*\tilde{\omega}$

Thm. (Poincaré – Lelong equation, Thuillier '05)

f meromorphic fn on X, $f \neq 0$

1) $[-\log|f|] \in D^{0,0}(X^{an})$ defines a current, i.e. $\omega \in A^{1,1}_c(X^{an})$, $\int_X -\log|f|\omega$ converges absolutely.

2) $d'd''[-\log|f|] - \delta_{\operatorname{div}(f)} = 0$ (PL equation)

Pf: 1) Omitted.

2) If $\operatorname{div} f = \sum_x n_x [x]$ then $\delta_{\operatorname{div}(f)}(\varphi) = \sum n_x \varphi(x)$

Need to check: $= \int_X -\log|f|\, d''d'(\varphi)$

V sst σ for $X \setminus |\operatorname{div} f| \longrightarrow$ get Σ and $|f|$ factors via τ_Σ by Thm 1)

Supp φ is cpt. $\Rightarrow$ we may refine V s.t. φ factors via τ_Σ

The argument works for any balanced + pw sm on Σ

- Auxiliary datum: choose orientation on each $e \in \Sigma$, $e = [x(e), y(e)]$, putting $y(e) := x$ for infinite edges, $x \in D \cup |\operatorname{div} f|$

Let φ'' on $\Sigma \setminus V$ be the second derivative, $F := -\log|f|$

$$\sum_{e\in\Sigma}\int_{x(e)}^{y(e)} F\cdot\varphi'' = \underbrace{\sum_{e\in\Sigma}\left[F\cdot\varphi'\right]_{x(e)}^{y(e)}}_{①} - \underbrace{\sum_{e\in\Sigma}\int_{x(e)}^{y(e)} n_e\cdot\varphi'}_{②}$$

int by parts

① $= 0$ by balancedness of φ', every non-open edge appears twice with opposite signs,

$\varphi'(x) = 0$ if $x \in D$ by cpt suppotedness of φ'

$\varphi'(x) = 0$ if $x \in |\operatorname{div} f|$ since φ constant on nbhd of canonical pts

$$② = - \sum_e [n_e \cdot \varphi]^{y(e)}_{x(e)} + 0$$ by the same argument as for ①

$$= - \sum_{x \in |\mathrm{div} f|} \mathrm{ord}_x(f)\varphi(x)$$ by balancedness of F, all vertices $x \in V$ cancel

$\varphi(x) = 0$ if $x \in D$, what remains is $x \in |\mathrm{div} f|$.

If $y(e) \in |\mathrm{div} f|$ then $n_e = \mathrm{ord}_x f$ by Thm. 5).

XVII Arakelov Chow groups

Soulé et al.: Lectures on Arakelov thy.

§1 Green currents / $\mathbb{C}$

X sm proj cx vty of dim d

On $X(\mathbb{C})$ we have: $A^{p,q}$ with $\partial: A^{p,q} \to A^{p+1,q}$ and $\bar\partial: A^{p,q} \to A^{p,q+1}$

Def. Endow $A^{p,q}$ w/ topology as before. Then $D_{p,q}(X) := \mathrm{Hom}^{\mathrm{cont}}(A^{p,q}(X), \mathbb{C})$ are the currents, $D^{p,q}(X) := D_{d-p,d-q}(X)$.

Ex. • $A^{p,q}(X) \hookrightarrow D^{p,q}(X)$, $\omega \mapsto [\omega] := \left[\eta \mapsto \int_{X(\mathbb{C})} \omega \wedge \eta\right]$

• $i: Y \hookrightarrow X$ closed subvty of codim p, $\delta_Y \in D^{p,p}(X)$, $\delta_Y(\eta) := \int_{Y^{ns}} i^*\eta$

where $Y^{ns} \subseteq Y(\mathbb{C})$ is the non-singular locus

Rmk. Thm of Hironaka: Given a proj vty Y and $Z \subseteq Y$ closed s.t. $Y \setminus Y^{ns} \subseteq Z$, there $\exists\, \tilde{Y} \xrightarrow{\pi} Y$ projective birational morphism s.t.

- $\tilde{Y}$ is non-singular
- $\pi|_{\pi^{-1}(Y \setminus Z)}$ is an iso
- the strict transform $\tilde{Z}$ of Z (i.e. closure of $\pi^{-1}(Z \cap Y^{ns})$) is a normal crossings divisor, i.e. locally on $\tilde{Y}(\mathbb{C})$, $\tilde{Z}$ is given by $\{z_1 \cdots z_k = 0\} \subseteq \mathbb{C}^n$ where z_i are the coordinates on $\mathbb{C}^n$.

Choosing $Z := Y \setminus Y^{ns} \subseteq Y$ we get $\int_{Y^{ns}} i^*\eta = \int_{\tilde{Y}} (i \circ \pi)^* \eta$. Thus δ_Y defines a current

Def. $d^c := \frac{1}{4\pi i}(\partial - \bar\partial)$

$$\Rightarrow dd^c = -\frac{1}{2\pi i}\partial\bar\partial$$

Def. Green current for a subvty $Y \subseteq X$ of codim p: $g \in D^{p-1,p-1}(X)$ s.t.

$dd^c g + \delta_Y = [\omega]$ for some $\omega \in A^{p,p}(X)$. (Again, $\partial, \bar\partial, d, d^c$ are def'd on $D^{\cdot\cdot}$ by duality.)

Thm. (X. Kähler) Any $Y \subseteq X$ has a Green current, and it is unique up to $A^{p-1,p-1}(X) + \partial D^{p-2,p-1}(X) + \bar{\partial} D^{p-1,p-2}(X)$.

PF: See book. □

This thm doesn't (yet) exist in the non-arch case.

§2 Metrised line bundles

$\mathcal{L}$ cx lb on X, i.e. loc free $\mathcal{C}^\infty = A^{0,0}$-module of rk 1

Def. Metric on $\mathcal{L}$: $\|\cdot\|^2 : \mathcal{L} \longrightarrow \mathcal{C}^\infty_{\geq 0}(-,\mathbb{R})$ s.t.
- $\|s\|^2(x) = 0 \iff s(x) = 0$
- $\forall f \in \mathcal{C}^\infty$: $\|fs\|^2 = |f| \cdot \|s\|^2$

Ex. • If $s_1, \dots, s_n \in \mathcal{L}(X)$ generate $\mathcal{L}$ then let

$$\|t\|^2(x) := \frac{|\psi(t)|^2}{\sum_i |\psi(s_i)|^2} \quad \text{where } \psi : \mathcal{L}|_U \xrightarrow{\cong} \mathcal{C}^\infty \text{ is a local trivialisation on } U \ni x.$$

In ptic, any cx lb has a metric.

We work with $\|\cdot\|^2$ instead of $\|\cdot\|$ b/c taking $\sqrt{\ }$ may not preserve smoothness.

• $(\mathcal{L}_1, \|\cdot\|_1), (\mathcal{L}_2, \|\cdot\|_2)$ metrised lbs $\Rightarrow$ $\mathcal{L}_1 \otimes \mathcal{L}_2$ is canonically metrised.

Now also assume that $\mathcal{L}$ is holomorphic, i.e. loc free $\mathcal{O}_X$-module.

Exc. $f \in \mathcal{O}_X(U)^\times \Rightarrow dd^c \log |f|^2 = 0$

Def. Curvature of $(\mathcal{L}, \|\cdot\|)$ or 1st Chern form: $c_1(\mathcal{L}, \|\cdot\|) \in A^{1,1}(X(\mathbb{C}))$ is the (1,1)-form locally given as $-dd^c \log \|s\|^2$ for s a holomorphic trivialising section of $\mathcal{L}$.

Rmk. $\Lambda^d c_1(\mathcal{L}, \|\cdot\|)$ is a (d,d)-form. If positive, it defines a measure on $X(\mathbb{C})$ (later).

These glue since $-dd^c \log \|f\|^2 = 0 \quad \forall f \in \mathcal{O}_X^\times$.

§3 Poincaré – Lelong /ℂ

Thm. $(\mathcal{L}, \|\cdot\|)$ holomorphic metrised lb on X, $s \neq 0$ meromorphic section.
- Then $-\log \|s\|^2 \in L^1(X)$ in ptic defines a current $[-\log \|s\|^2] \in D^{0,0}(X)$.
- It is a Green current for div(s), i.e. $dd^c[-\log \|s\|^2] + \delta_{\mathrm{div}(s)} = [c_1(\mathcal{L}, \|\cdot\|)]$ in $D^{1,1}(X(\mathbb{C}))$ where if $\mathrm{div}(s) = \sum n_Z Z$ then $\delta_{\mathrm{div}(s)} = \sum n_Z \delta_Z$.

PF: Claim on L^1: Using a partition of unity and resolution of singularities with $Z = \mathrm{Supp}\, \mathrm{div}(s)$, we reduce to the following.

| Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{C}^n)$. Then $\int_{\mathbb{C}^n} \varphi \cdot \log \|z_1 \cdots z_k\|^2$ is abs convergent.

With log being additive, we reduce to the case $k=1$.

Then $\mathrm{Vol}(\{|z_1| < \varepsilon\} \cap \mathrm{Supp}\, \varphi) = O(\varepsilon^2)$ for $\varepsilon \to 0$, so it follows that

$$\int_{|z_1|<\varepsilon} \left|\log \|z_1\|^2 \cdot \varphi\right| = \mathcal{O}\left(\varepsilon^2 \cdot \log \varepsilon\right) \text{ for } \varepsilon \to 0.$$ This shows the L^1-property. ✓

We now prove the PL equation part.

Nts $\forall \omega \in A^{d-p,d-p}(X)$. $\displaystyle\int_X -\log \|s\|^2 \, dd^c\omega + \int_{\mathrm{div}(s)} \omega = \int_{X(\mathbb{C})} c_1(\mathcal{L}, \|\cdot\|) \wedge \omega$

- Apply resol of sing for X with $Z := \mathrm{Supp}\ \mathrm{div}(s)$, get $\pi : \tilde{X} \longrightarrow X$, $\tilde{Y} \subseteq \tilde{X}$ normal crossing divisor := strict transform of $\mathrm{div}(s)$.

  This π is an iso away from some subset of codim ≥ 1, which thus has Lebesgue measure 0 $\Rightarrow$ integrals over X can be computed over $\tilde{X}$.

  The same goes for $\int_{\mathrm{div}(s)} \omega$ because "$\pi|_{\tilde{Y}} : \tilde{Y} \longrightarrow \mathrm{div}(s)$" is iso away from codim ≥ 1.

- Decompose ω with partition of unity $\longrightarrow$ we may work on charts. $\Rightarrow$ Wma $\mathrm{div}(s)$ to be a normal crossing.

<u>Case 1.</u>: $\mathrm{Supp}\ \omega \cap \mathrm{div}(s) = \emptyset$

$\Rightarrow$ $\mathrm{div}\ s$ is holomorphic section of $\mathcal{L}|_{X \setminus \mathrm{div}(s)}$

By def.: $-dd^c \log \|s\|^2 \big|_{X \setminus \mathrm{div}(s)} = c_1(\mathcal{L}, \|\cdot\|)$

<u>Case 2</u>: $\mathrm{Supp}\ \omega \cap \mathrm{div}(s) \neq \emptyset$, $n := d$

In a local chart $(0,\dots,0) \in U \subseteq \mathbb{C}^n$, $\mathcal{L} = \mathcal{O}_U$ normal lb, $\underbrace{\|1\|^2}_{\psi :=}$ smooth function giving metric, $s = z_1^{m_1} \cdots z_n^{m_n}$ with $m_i \in \mathbb{Z}$.

$$\Rightarrow \quad -\log \|s\|^2 = -\log \psi - \sum_{i=1}^{n} m_i \log |z_i|^2$$

Test form $\omega \in A_c^{n-1,n-1}(U)$: again $\int dd^c \log \psi \wedge \omega = \int c_1(\mathcal{L}, \|\cdot\|) \wedge \omega$ by defn.

By linearity, it remains to check $\displaystyle\int_U \log |z_1|^2 \wedge dd^c\omega = \int_{z_1=0} \omega$

$$\mathrm{LHS} = \lim_{\varepsilon \to 0} \int_{|z_1|>\varepsilon} \log |z_1|^2 \wedge dd^c\omega$$ since the integral over the complement $\to 0$ when $\varepsilon \to 0$ as above

$$= -\underbrace{\lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} \log |z_1|^2 \wedge d^c\omega}_{\mathcal{O}(\varepsilon \log \varepsilon)} - \lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} d \log |z_1|^2 \wedge d^c\omega$$ by Stokes + int. by parts

$$= \lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} d^c \log |z_1|^2 \wedge d\omega$$ by $d^c \log |z_1|^2 \wedge d\omega = -d \log |z_1|^2 \wedge d^c\omega$ (direct computation)

$$= \lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} d^c \log |z_1|^2 \wedge \omega + \underbrace{\lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} dd^c \log |z_1|^2 \wedge \omega}_{0 \text{ by Exc.}}$$ by Stokes + ibp

Exc.: $d^c \log |z_1|^2 = \dfrac{1}{2\pi} d \arg(z_1) = \displaystyle\int_{z_1=0} \omega$.

§4 Arakelov Chow groups

X regular flat projective scheme $/\mathrm{Spec}\,\mathbb{Z}$

$F_\infty : X(\mathbb{C}) \longrightarrow X(\mathbb{C})$ complex conjugation operator

$A^{p,p}(X) := \{ \omega \in A^{p,p}(X(\mathbb{C})) \mid (-1)^p F_\infty^* \omega = \omega \}$

$D^{p,p}(X) := \{ T \in D^{p,p}(X(\mathbb{C})) \mid (-1)^p F_\infty^* T = T \}$

Def. Green current for a cycle $Z = \sum_\alpha Z_\alpha \in Z^p(X)$: $g \in D^{p-1,p-1}(X)$ s.t. $\exists \omega_{(Z,g)} \in A^{p,p}(X)$,

$dd^c g + \delta_Z = [\omega_{(Z,g)}]$.

Here $Z^p(X) := \bigoplus\limits_{\substack{Y \subseteq X \\ \text{int subscheme} \\ \text{of codim } p}} \mathbb{Z} \cdot Y$

Ex. • $f \in \mathcal{O}_{X,\eta}^\times$ meromorphic invertible fn. on X. Then by PL: $[-\log |f|^2]$ is a Green current for $\mathrm{div}(f)$.

• More generally, $Y \xhookrightarrow{i} X$ codim $p-1$ integral subscheme, $f \in \mathcal{O}_{Y,\eta}^\times$, $i_* \mathrm{div}(f) \in Z^p(X)$.

Y may be singular $\Rightarrow$ PL cannot be applied directly.

Let $\tilde{Y}_\mathbb{C}$ be a desingularisation of $Y \times \mathrm{Spec}\,\mathbb{C}$. wrt. $|\mathrm{div}(f)|$.

PL: $[-\log |\tilde{f}|^2]$ is a Green current for $\mathrm{div}(\tilde{f})$

$(i \circ \pi_{\tilde{Y}_\mathbb{C} \to Y})_* \left[-\log |\tilde{f}|^2 \right] :=$ dual to pullback $(i \circ \pi)^* : A^{p-1,p-1}(X(\mathbb{C})) \to A^{p-1,p-1}(\tilde{Y}_\mathbb{C}(\mathbb{C}))$

This is a Green current for $\mathrm{div}(f)$.

• If $u \in D^{p-2,p-1}(X(\mathbb{C}))$, $v \in D^{p-1,p-2}(X(\mathbb{C}))$ then $dd^c(\partial u + \bar{\partial} v) = 0$

$\Rightarrow \partial u + \bar{\partial} v$ is a Green current for $0 \in Z^p(X)$

Def. $\hat{Z}^p(X) := \{ (Z, g_Z) \mid Z \in Z^p(X),\ g_Z \text{ Green current for } Z \}$ group of arithmetic cycles on X

$\hat{B}^p(X) := \langle \partial D^{p-2,p-1}(X),\ \bar{\partial} D^{p-1,p-2}(X),\ \{ \underbrace{(\mathrm{div}\, f, [-\log |f|^2])}_{\text{principal arithmetic cycles}} \mid Y \subseteq X \text{ codim } p-1 \text{ int subsch},\ f \in \mathcal{O}_{Y,\eta}^\times \} \rangle$

Arakelov Chow group: $\widehat{CH}^p(X) := \hat{Z}^p(X) / \hat{B}^p(X)$

Ex. $X \longrightarrow \mathrm{Spec}\,\mathbb{Z}$ relatively 1-dim.

Then $\widehat{CH}^0(X) \simeq \mathbb{Z}$: 0-cycles are of the form nX, $n \in \mathbb{Z}$, $g_Z \in D^{-1,-1}(X) = 0$

$\widehat{CH}^1(X) \simeq \widehat{\mathrm{Pic}}^1(X) := \{ (\mathcal{L}, \|\cdot\|) \mid \mathcal{L} \text{ lb on } X,\ \|\cdot\| \text{ metric on } \mathcal{L}_\mathbb{C} \} /\sim$

$(Z, g_Z) \longmapsto \left(\mathcal{O}(Z),\ -\log \|1\|^2 - g_Z \right)$

$(\mathrm{div}\, s, -\log \|s\|^2) \longleftarrow (\mathcal{L}, \|\cdot\|)$

$$\widehat{CH}^2(X) = \left\{\left(\sum_P n_P P,\ g \in D^{1,1}(X)\right)\right\} / \text{relations} \qquad P:\ \text{point}$$

Arakelov degree: $\widehat{CH}^2(X) \longrightarrow \mathbb{R}$ This map factors over $\widehat{CH}^2(X)$.

$$\left(\sum n_P P, g\right) \longmapsto \sum_P n_P \log|\mathbb{F}_P| + \underbrace{\int_{X(\mathbb{C})} g}_{:= g(1)}$$

Here we used that • if $C \hookrightarrow X$ is a curve and $f \in \mathcal{O}^\times_{C,\eta}$, $\operatorname{div} f = \sum_P n_P P$

then $\sum_P n_P \log|\mathbb{F}_P| = 0$.

- Product formula for number fields, if $C \to \operatorname{Spec} \mathbb{Z}$ flat for function field of $C \to \operatorname{Spec} \mathbb{F}_p$ for some p
- similar $\widehat{\deg}(0, \partial u + \bar\partial v) = 0$

XVIII Non-archimedean metrised line bundles

13.12.2018

XIX: Study of the case of curves, in ptic def. $\hat{Z}$ P.ian

XX: Higher dimensional case

We assume $K = \bar{K}$, $X/\operatorname{Spec} K$ sep oft

§1 Approximating continuous functions on X^{an}

Ref.: CL–D § 3.3

The functions coming from models are only pw smooth.

Such approximations are quite common in the field.

Recall. X^{an} is loc cpt and T2

Lemma. $X = \operatorname{Spec} A$, $x \neq y \in X^{an}$. Then $\exists \varphi$ smooth s.t. $\varphi_x \equiv 0$, $\varphi_y \equiv 1$,

i.e. $\exists x \in \mathcal{U}_x$, $y \in \mathcal{U}_y$ nbhds s.t. $\varphi|_{\mathcal{U}_x} \equiv 0$ and $\varphi|_{\mathcal{U}_y} \equiv 1$.

Pf: $\exists f \in A$: $|f(x)| \neq |f(y)|$ by def of X^{an}

Consider $\varphi := \psi \circ |f|$ where $\psi: \mathbb{R} \to \mathbb{R}$ is smooth s.t. $\psi|_{|f(x)|} \equiv 0$ and $\psi|_{|f(y)|} \equiv 1$.

Cor. $X = \operatorname{Spec} A$, $K \subseteq X^{an}$ cpt, $y \notin K$. Then $\exists \varphi$ smooth and $K \in \mathcal{U}$ open s.t. $\varphi|_{\mathcal{U}} \equiv 0$.

Pf: $\forall x \in K$ find φ^x as in Lemma, $x \in \mathcal{U}_x$ s.t. $\varphi^x|_{\mathcal{U}_x} \equiv 0$.

These $\mathcal{U}_x$ cover K, K is cpt $\Rightarrow \exists x_1, \dots, x_n$ s.t. $K \subseteq \bigcup_{i=1}^n \mathcal{U}_{x_i}$.

Then $\varphi := \prod_{i=1}^n \varphi^{x_i}$ does the job.

Cor. X variety, $x \in \mathcal{U} \subseteq X^{an}$ open. Then $\exists \varphi$ smooth s.t. $\varphi_x \equiv 1$ and $\operatorname{Supp} \varphi \subseteq \mathcal{U}$ cpt.

Pf: Choose $\operatorname{Spec} A \underset{\text{open}}{\subseteq} X$ s.t. $x \in (\operatorname{Spec} A)^{an}$

Let $x \in K \subseteq \mathcal{U} \cap (\operatorname{Spec} A)^{an}$ be a compact nbhd. (i.e. $x \in V \subseteq K$ for V open)

Let $x \in K' \subseteq \mathring{K}$ cpt nbhd. By Cor.: $\exists \varphi$ smooth, $\varphi_x \equiv 1$ and $\varphi \equiv 0$ on a nbhd of $K \setminus (K')^\circ$

Then the 0 function on $X^{an}\setminus K$ and φ on $\mathring{K}$ glue to a suitable function. $\square$

Cor. $K \subseteq U \subseteq X^{an}$ cpt in open. Then $\exists \varphi : X^{an} \to [0,1]$ smooth s.t. $\operatorname{Supp} \varphi \subseteq U$ and cpt., and $\varphi \equiv 1$ on a nbhd of K.

Pf: $\forall x \in K$ find φ^x for U as in the prev Cor. Replace φ^x by $(\varphi^x)^2 \Rightarrow$ wma $\varphi^x \geq 0$.

Again $U_x \ni x$ s.t. $\varphi^x|_{U_x} = 1$, and K can be covered by $U_{x_1}, \ldots, U_{x_n}$

$\varphi' := \sum_{i=1}^{n} \varphi^{x_i} \Rightarrow \varphi'|_K \geq 0$, $\operatorname{Supp} \varphi' \subseteq U$ and cpt

$\varphi := \psi \circ \varphi'$ where $\psi : \mathbb{R} \to \mathbb{R}$ smooth, $\psi(0) = 0$, $\psi(\varphi'(K)) = \{1\}$. This φ does the job. $\square$

Prop. $U \subseteq X^{an}$ open, $f : U \to \mathbb{R}$ continuous, $\operatorname{Supp} f$ cpt. Then $\forall \varepsilon > 0$ $\exists \varphi \in C^\infty(U)$ s.t.

$|\varphi - f|_{sup} \overset{def}{=} \sup\{|\varphi(x) - f(x)| \mid x \in U\} < \varepsilon$.

Recall the Stone-Weierstrass Thm: M cpt T2, $P \subseteq C^0(M, \mathbb{R})$ sub-$\mathbb{R}$-algebra s.t.

- $\forall x, y \in M$, $x \neq y$: $\exists \varphi \in P$: $\varphi(x) \neq \varphi(y)$
- $\forall x \in M$ $\exists \varphi \in P$: $\varphi(x) \neq 0$

} these may be equivalent, depending on the def. of "sub-$\mathbb{R}$-alg."

Then P is dense in $C^0(M, \mathbb{R})$ wrt. $|\cdot|_{sup}$.

PF OF PROP: $X' := X^{an} \cup \{\infty\}$ 1-pt compactification,

$P :=$ generated by constant functions and $C_c^\infty(X^{an})$. Note: $\forall \varphi \in P$ is a sum const $+ \varphi'$ for $\varphi' \in C_c^\infty(X^{an})$

Apply SW to X' and P.

$\Rightarrow$ given an f, $\exists c + \varphi' \in P$ s.t. $|f - c - \varphi'|_{sup} < \frac{\varepsilon}{2}$

$f(\infty) = \varphi'(\infty) = 0 \Rightarrow |c| < \frac{\varepsilon}{2} \Rightarrow |f - \varphi'|_{sup} < \varepsilon$

$\varphi := \varphi' \circ \psi$ where ψ is as in the prev Cor applied to $\operatorname{Supp} f \subseteq U$,
(i.e. $\operatorname{Supp} \psi$ cpt., $\subseteq U$, $\psi \equiv 1$ on a nbhd of $\operatorname{Supp} f$) $\square$

Application

Prop. (G-K, Prop 6.13) $U \subseteq X^{an}$, $\alpha \in A_c^{d,d}(U)$. Then $[\alpha] : A_c^{d,d}(U) \to \mathbb{R}$

$$\varphi \longmapsto \int_U \varphi \alpha$$

is continuous wrt $|\cdot|_{sup}$ on $A_c^{0,0}(U)$. @

In ptic, it extends (uniquely) to $C_c^0(U, \mathbb{R})$.

@ More precisely, $\forall K \subseteq U$ cpt: $[\alpha]\big|_{C_K^\infty(U)}$ is continuous wrt $|\cdot|_{sup}$ and extends to $C_K^0(U)$. Here C_K means: has support in K.

Recall. Riesz-Markov-Kakutani Representation Thm.: M loc cpt T2 space. Then

$\{\ell : C_c^0(M, \mathbb{R}) \to \mathbb{R} \mid$ cont wrt. $|\cdot|_{sup}$ on all $C_K^0(M, \mathbb{R})$ for $K \subseteq M$ cpt$\} \xleftrightarrow{1:1}$

$\longleftrightarrow \{$ signed Radon measures $\mu = \mu_0 - \mu_1$ (i.e. μ_0, μ_1 Radon measures)$\}$

given by $\int_M \cdot \, d\mu \longleftarrow \mu$.

<u>Cor.</u> $\forall \alpha \in A_c^{d,d}(U)$ $\exists!$ μ_α signed Radon measure s.t. $\forall f \in C_c^\infty(U)$: $\int_{|X^{an}|} f \, d\mu_\alpha = \int_{X^{an}} f \alpha$

<u>Def.</u> $f: U \to \mathbb{R}$ cont. Then $[f]: A_c^{d,d}(U) \longrightarrow \mathbb{R}$

$$\alpha \longmapsto \int_{X^{an}} f \, d\mu_\alpha$$

<u>Prop.</u> (G–K Prop. 6.16) $[f]$ is continuous and hence defines a current $\in D^{0,0}(U)$.

Upshot: we have $C^0(U,\mathbb{R}) \longrightarrow D^{0,0}(U)$ continuous.

§2 Metrised line bundles and Poincaré–Lelong

Consider $\mathcal{L}$ a lb on X, i.e. rk 1 loc free $\mathcal{O}_X$-module.

Have a universal $(X^{an}, \mathcal{O}_{X^{an}}) \xrightarrow{\pi} (X, \mathcal{O}_X)$ map of lrs

$\Rightarrow$ get $\mathcal{L}^{ad} := \pi^* \mathcal{L}$ lb on X^{ad}, which then defines $\mathcal{L}^{an} := q_* \mathcal{L}$ on X^{an}

<u>Def.</u> Continuous metric on $\mathcal{L}$: $\|\cdot\|: \mathcal{L}^{ad} \longrightarrow C^0(-, \mathbb{R}_{\geq 0})$ s.t.

1) $\|s\|(x) = 0 \iff s(x) = 0$
2) $\|f \cdot s\| = |f| \cdot \|s\|$ $\forall f \in \mathcal{O}_{X^{ad}}$

<u>Def.</u> $\|\cdot\|$ is smooth / pw. smooth / pw linear / ... if $\forall s$ invertible section: $\|s\|$ is smooth/... .

<u>Def.</u> $f \in C^0(W, \mathbb{R})$, $W \subseteq X^{an}$ is piecewise linear if locally $\exists$ (U, φ, V) trop charts s.t. $f|_V = \psi \circ \mathrm{trop}_\varphi|_V$ where $\psi: |\mathcal{C}| \to \mathbb{R}$ is Γ-affine on all $\sigma \in \mathcal{C}$ and continuous on $|\mathcal{C}|$, and $\mathrm{trop}_\varphi(V) \subseteq \mathcal{C}$ is a Γ-rational polyhedral complex.

Recall: $\Gamma = \log(|K^\times|)$

Let $(\mathcal{L}, \|\cdot\|)$ be a metrised lb. notice the brackets!

<u>Def.</u> Curvature form: $[c_1(\mathcal{L}, \|\cdot\|)] \in D^{1,1}(X^{an})$ locally defined as $[c_1(\mathcal{L}, \|\cdot\|)] = d'd''[-\log \|s\|]$ where s is a local invertible section.

<u>Lemma.</u> $\forall f \in \mathcal{O}_{X^{an}}(U)^\times$: $d'd''(-\log|f|) = 0$

Pf: We may work locally, i.e. inside $K \subseteq U^{an}$ cpt.

When $K \subseteq (\mathrm{Spec}\, A)^{an}$ for $\mathrm{Spec}\, A \subseteq X$

Approximation: $\exists f^{alg} \in A$ s.t. $|f|\big|_K = |f^{alg}|\big|_K$

$\Rightarrow K \subseteq \left(\mathrm{Spec}\, A[f^{alg,-1}]\right)^{an} =: W$

Locally on K, we may extend f^{alg} to a tropical chart $(f^{alg}, f_1, \dots, f_r): \mathrm{Spec}\, B \overset{@}{\hookrightarrow} \mathbb{G}_m^{r+1}$

Then $d'd''[-\log|f^{alg}|] = [-d'd'' \log|f^{alg}|]$. on the tropical chart @, represented by $d'd''(x_0) = d'(d'' x_0) = 0$ where x_0 is the 0^{th} coordinate function.

Rmk. If $\|\cdot\|$ is smooth then $[c_1(\mathcal{L}, \|\cdot\|)] \in A^{1,1}(X^{an})$

Thm. (G-K, Thm. 7.2 + Cor. 7.8) $(\mathcal{L}, \|\cdot\|)$ metrised lb, s merom section of $\mathcal{L}$, nowhere trivial.

Then $d'd''[-\log \|s\|] + \delta_{div(s)} = [c_1(\mathcal{L}, \|\cdot\|)]$ in $D^{1,1}(X^{an})$ □.

PO, the proof is quite complicated.

Rmk. As for curves, $-\log\|s\|$ defines a current since $\forall \alpha \in A_c^{d,d}(X^{an})$:

$\operatorname{supp}(\alpha) \cap \operatorname{div}(s)^{an} = \emptyset$.

§3 Metrics induced from models

Def./Construction. Let $\mathcal{L}^+$ be a locally free $\mathcal{O}^+_{X^{ad}}$-module on X^{ad} of rk 1.

Then $\mathcal{L} := \mathcal{O}_{X^{ad}} \otimes_{\mathcal{O}^+_{X^{ad}}} \mathcal{L}^+$ has a canonical metric:

$\|s\| = 1$ for s trivialising $\mathcal{L}^+$

Gluing / well-def since $\forall f \in \mathcal{O}^{+,\times}_{X^{ad}}$: $\|f\| = 1$.

Let $\mathfrak{X}/\operatorname{Spf} \mathcal{O}_K$ be an admissible formal scheme, $\mathcal{L}$ lb on $\mathfrak{X}$

Then we have $sp: (\mathfrak{X}^{ad}_\eta, \mathcal{O}^+_{\mathfrak{X}^{ad}_\eta}) \longrightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$

Get $\mathcal{L}^{ad,+} := sp^*\mathcal{L}$ loc free of rk 1 $\mathcal{O}^+_{\mathfrak{X}^{ad}_\eta}$-module

Then $\mathcal{L}^{ad} := \mathcal{O}_{X^{ad}} \otimes_{\mathcal{O}^+_{X^{ad}}} \mathcal{L}^{ad,+}$ is metrised.

Def. Formally metrised: $(\mathcal{L}, \|\cdot\|)$ isomorphic to such a metrised lb.

Fact. Any $\mathcal{L}/X \rightsquigarrow \mathcal{L}^{ad}/X^{ad}$ has a formal model, so in ptic a formal metric.

Prop. $(\mathcal{L}, \|\cdot\|)$ metrised lb. TFAE:

(1) $\|\cdot\|$ is formal

(2) $\|\cdot\|$ is pw. linear

(3) $\exists X^{ad} = \bigcup_i U_i$ and $s_i \in \mathcal{L}(U_i)$ trivialising s.t. $\|s_i\| = 1$.

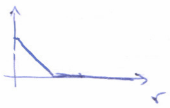
Technical difficulty: formal metrics are usually not smooth

$\rightarrow$ $[c_1(\mathcal{L}, \|\cdot\|)]$ is only a current.

Two solutions to this:

- "classical" solution: CL-D, approximate the metric by smooth metrics.
  Then one can define $\mu_{\wedge^d c_1(\mathcal{L}, \|\cdot\|)}$ but not $\wedge^d c_1(\mathcal{L}, \|\cdot\|)$.
- New approach, G-K: enlarge $A^{p,q}$ to "(p,q)-f-forms" admitting $\wedge$-product
  $\rightarrow$ $c_1(\mathcal{L}, \|\cdot\|)$ is a 1-1-f-form.

<u>Exc.</u> Write down a formal model of $(\mathbb{D}, \mathcal{O}_{\mathbb{D}})$, inducing $\|\cdot\|$ formal metric on $\mathcal{O}_{\mathbb{D}}$ s.t.

$\|1\| =$  up to translation.

Idea: take the circle, blow up s.t. we get $\mathbb{P}^1$.

XIX Thuillier's theory for curves

17.12.201

Ref.: Thesis of Thuillier

<u>Recall.</u> $\mathfrak{X}/\mathrm{Spf}\, \mathcal{O}_K$ adm formal, $\mathcal{L}$ lb on $\mathfrak{X}$ $\Rightarrow$ $\mathcal{L}^{ad}$ on $\mathfrak{X}^{ad}$ can metrised, $\|\cdot\|$

Given $s \in \mathcal{L}^{ad}(\mathcal{U})$, $\mathcal{U} \subseteq \mathfrak{X}^{ad}_\eta$ open, $x : \mathrm{Spa}(L, L^+) \to \mathcal{U}$,

x defines $\mathrm{Spf}\, L^+ \xrightarrow{\tilde{x}} \mathfrak{X}$, $\tilde{x}^* \mathcal{L} = M^+$ is an L^+-mod free of rk 1, choose an iso $M^+ \simeq \mathcal{L}^+$

$\Rightarrow s(x) = x^* \mathcal{L}^{ad} = M^+\left[\frac{1}{\pi}\right] \simeq L$ and $\|s(x)\| = |$ of $s(x)$ in $L|$

<u>Ex.</u> If s comes from a global section $\mathcal{L}(\mathfrak{X})$ then $\|s(x)\| = 1$ if $sp(x) \in \mathfrak{X}_k \setminus \{s = 0\}$

<u>Prop.</u> (G-K, Prop 8.11) $X/\mathrm{Spec}\, K$ sep oft, $\mathcal{L}$ lb on X. Then $\|\cdot\|$ on $\mathcal{L}$ is formal iff pw lin

<u>Recall.</u> $(\mathcal{L}, \|\cdot\|) \longmapsto c_1(\mathcal{L}, \|\cdot\|) \in D^{1,1}(X^{an})$

cont metric

$\hookrightarrow$ Need to enlarge / modify to get $\in A^{1,1}(X^{an})$

§1 Thuillier forms

X sm sep curve $/K$

<u>Def.</u> $TA^{0,0} \subseteq C^0(-, \mathbb{R})$ is a sheaf gen by $\psi \cdot \tau_\Sigma$ where ψ is cont on Σ and linear on the edges. I.e. $TA^{0,0}$ is the sheaf of pw lin functions.

<u>Rmk.</u> If $\varphi \in TA^{0,0}_c(\mathcal{U}) \subseteq TA^{0,0}_c(X^{an})$ then $\exists V$ sst vs, $\exists \psi$ s.t. $\varphi = \psi \cdot \tau_{\Sigma(X,V)}$

<u>Idea</u> to define $TA^{1,1}$: as $TA^{0,0} \subseteq D^{0,0}$, compute $d'd''[f]$ for $f \in TA^{0,0}$.

For $\varphi \in A^{0,0}_c(X^{an})$ smooth, $f = \psi \cdot \tau_\Sigma$

$$\big(d'd''[f]\big)(\varphi) = [f]\big(d'd''\varphi\big) = \sum_{\substack{e \subseteq \Sigma \\ \text{edge}}} \int_e f \cdot \psi'' \, d'x \wedge d''x$$

integration by parts

$$= \underbrace{\sum_{\substack{e \subseteq \Sigma \\ \text{edge}}} \left([f \psi']_{x(e)}^{y(e)} - \int_e f' \cdot \psi' \, dx \right)}$$

integration by parts

$$= \underbrace{\sum_{\substack{x \in \Sigma \\ \text{vertex}}} f(x) \cdot \underbrace{\sum_{v \in T_x} d_v \psi(x)}_{0}}_{0} - \underbrace{\sum_{e} \left(-[f' \psi]_{x(e)}^{y(e)} \right.} + \underbrace{\left. \int_e f'' \psi \, dx \right)}_{0 \text{ since } f \text{ lin}}$$

$$= \sum_{\substack{x \in \Sigma \\ \text{vertex}}} \psi(x) \sum_{v \in T_x} d_v f(x)$$

This need not be zero. (we have no balancedness condition)

$$\Rightarrow \quad d'd''[f] = \sum_{x\,(2)} \left(\sum_{v \in T_x} d_v f(x) \right) \delta_x \qquad @$$

This motivates the following

Def. $\underline{TA^{1,1}} := \bigoplus_{x\,(2)} \mathbb{R}\,\delta_x = \left\{ \sum_{x\,(2)} r_x \delta_x \ \text{loc fin} \right\} \subseteq D^{1,1}$

$\underline{d'd''} : TA^{0,0} \longrightarrow TA^{1,1}$ def'd by @

$\underline{\mathcal{H}} := \operatorname{Ker} d'd''$ harmonic functions

There are exact sequences:

$$0 \longrightarrow \mathcal{H} \longrightarrow TA^{0,0} \xrightarrow{d'd''} TA^{1,1} \longrightarrow 0,$$

$$\mathcal{O}^{\times}_{X^{an}} \xrightarrow{f \mapsto \log|f|} \mathcal{H} \xrightarrow{(*)} \bigoplus_{x\,(2)} \operatorname{Re}^{\circ}(C_x) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow 0$$

where $(*)$: $\varphi \longmapsto \left[x \mapsto \sum_{v \in C_x(k)} d_v \varphi(x) \cdot x \right]$ finite sum, $= 0$ if $C_x \simeq \mathbb{P}^1_k$

We have $\underline{\int} : TA^{1,1}_c(X^{an}) \longrightarrow \mathbb{R}, \quad \sum_{\text{fin}} r_x \delta_x \longmapsto \sum r_x$

Rmk. G-K also enlarge $A^{0,0}$ and $A^{1,1}$.

Thus one eventually gets: pw smooth functions in deg $(0,0)$,

δ_x for x (2),

formal $\sum_e \omega_e$ for ω_e (1,1)-form, no gluing condition.

Def. $Z \in \operatorname{Div}(X)$ divisor. Then a Green current (Green function) for Z is a smooth function $g : X^{an} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ cont. s.t. $\exists\, \omega_{(Z,g)} \in TA^{1,1}$ s.t. $d'd''[g] + \delta_Z = [\omega_{(Z,g)}]$

Equivalently, $g|_{X^{an} \setminus Z} \in TA^{0,0}(X^{an} \setminus Z)$ and $\forall U \subseteq X$ open; $s \in \mathcal{O}_X(U)$: $\operatorname{div}(s) = Z|_U$, then $g|_U = -\log|s| + \varphi$ for some $\varphi \in TA^{0,0}(U)$. (The equivalence is "not entirely obvious".)

Rmk. Given g_1, g_2 GF for Z, $g_1 - g_2|_{X^{an} \setminus Z} \in TA^{0,0}(X^{an} \setminus Z)$ extends to a function $g_1 - g_2 \in TA^{0,0}(X^{an})$.

If $\omega_{(Z,g_1)} = \omega_{(Z,g_2)}$ then $g_1 - g_2 \in \mathcal{H}(X^{an})$

If X is projective or, more generally, $\forall X_0 \subseteq X$ conn comp. : $\#|\hat{X}_0 \setminus X_0| \leq 1$ then $\mathcal{H}(X^{an}) = \mathbb{R}^{\pi_0(X)}$.

Then for $\omega_{(Z,g)}$ fixed, g is unique up to const.

Rmk. If $z \in |\operatorname{Supp} Z|$. let $0 \simeq z \in \mathbb{D}^{an}$, $\mathbb{D}^{an} \cap |\operatorname{Supp} Z| = \{z\}$. Then $g = n_z \log|t| + \text{const}$.

§2 Arakelov groups

$X/\mathbb{Q}$ smooth projective curve

Study flat proj models over $U \subseteq \operatorname{Spec} \mathbb{Z}$.

Existence: $X = V_+(f_1, \dots, f_n) \subseteq \mathbb{P}^N$, $U = \operatorname{Spec} \mathbb{Z}\left[(\text{products of all denominators of all coeffs of } f_i)^{-1}\right]$

$\Rightarrow \mathcal{X} := V_+(f_i) \subseteq \mathbb{P}^N_U$ Taking flat reduced closure of $X \subseteq \mathcal{X}$, we get flat proj $\mathcal{X}|_U$.

$\downarrow$ model

U

Prop. 1) $\forall \mathcal{X}|_U$ $\exists U' \subseteq U$ s.t. $\mathcal{X}|_{U'} \to U'$ smooth (in ptic reg & sst)

2) Given $\mathcal{X}_1, \mathcal{X}_2|_U$ $\exists U' \subseteq U$ open s.t. id_X lifts to $\mathcal{X}_1|_U \simeq \mathcal{X}_2|_U$.

3) If $\mathcal{L}$ is a lb on X, $s \in \mathcal{L}(X)$ (meromorphic) section, and $\mathcal{X}|_U$ is given then $\exists U' \subseteq U$ open, $\tilde{\mathcal{L}}$ on $\mathcal{X}|_{U'}$ model of $\mathcal{L}$, $\tilde{s} \in \tilde{\mathcal{L}}(\mathcal{X}|_{U'})$ (mero.) extending s.

Pf: 1) non-smooth-locus $\mathcal{X}^{ns} \subseteq \mathcal{X}$ closed, $\mathcal{X}^{ns} \cap X = \emptyset \Rightarrow \mathcal{X}^{ns} \subseteq \bigcup_{\text{fin}}$ closed fibres

2) The generic iso id_X extends except possibly for $\underbrace{\text{codim } 2}_{\subseteq \bigcup_{\text{fin}} \text{ cl fibres}}$

3) similar but more involved.

Def. X as above. A Thuillier–Arakelov divisor on X is $(D, (g_p)_p)$ where

- $D \in \operatorname{Div}(X)$
- $\forall p$: g_p is a Green current for D in the sense defined in XIX for $p < \infty$, and as defined before for $p = \infty$

s.t. for some (for any) $\mathcal{X}|_U$ flat proj model and $\mathcal{L}$ model of $\mathcal{O}(D)$ and s meromorphic section $\mathcal{O} \to \mathcal{O}(D)$:

for almost all p, $g_p = -\log \|s\|_p$ where $\|\cdot\|_p$ is the formal metric on $\mathcal{L}$ from the model

Notation. $\mathbb{C}_p := \begin{cases} \widehat{\overline{\mathbb{Q}}}_p & \text{for } p < \infty \\ \mathbb{C} & \text{for } p = \infty \end{cases}$ $\quad X_p := X \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{C}_p$, $\quad X_p^{an} := \begin{cases} X_p^{an} & p < \infty \\ X_p(\mathbb{C}) & p = \infty \end{cases}$

For $D \in \operatorname{Div}(X)$, let $D_p := D \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{C}_p \in \operatorname{Div}(X_p)$

Rmk. If $(\mathcal{X}|_U, \mathcal{L}, s)$ as in the Def., then for almost all $p < \infty$:

1) $\mathfrak{X}_p := \left(\mathcal{X} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathcal{O}_{\mathbb{C}_p}\right)^\wedge_p$ is smooth, in ptic sst

2) $\mathcal{D} := \overline{D}^x$ flat closure (if e.g. $D: \operatorname{Spec} M \to X$ then $\mathcal{D} \simeq \operatorname{Spec} \mathcal{O}$, $\mathcal{O} \subseteq M$ some order)

$\mathfrak{D}_p :=$ div in $\mathfrak{X}_p$, $\mathfrak{D}_p \to \operatorname{Spf} \mathcal{O}_{\mathbb{C}_p}$ is smooth, i.e. $\simeq \coprod \operatorname{Spf} \mathcal{O}_{\mathbb{C}_p}$

3) $D_p \xrightarrow{\simeq} \mathrm{div}(s)_p$

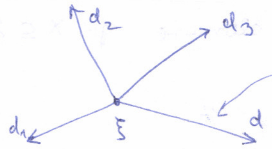
At these p, for each $X^\circ \subseteq X_p$ conn comp with closure $\mathfrak{X}_p^\circ$:

$V(\mathfrak{X}_p^\circ) = \{\xi\} \subseteq X^\circ$ is a sst vs by 1)

By 2), it is also a sst vs for $X^\circ \setminus (D_p \cap X^\circ)$

By 3) we get: $-\log \|s\|_p(\xi) = 0$, $\quad -\log \|s\|_p \big|_{X^\circ \setminus (D \cap X^\circ \cup \xi)}$

$\Sigma(X^\circ \setminus (D \cap X^\circ), \{\xi\})$:



$-\log \|s\|_p = $ coeff of d in $D = \mathrm{dist}(\xi, x)$

Then $d'd''[-\log \|s\|_p] = \deg(D \cap X^\circ)\,\delta_\xi$

Ex. $f \in \mathcal{O}^\times_{X,\eta}$ Then $(\mathrm{div}\, f, (-\log \|f\|_p)_p) \in \widehat{TZ}^1(X)$

Ex. $D = 0$. Then the GFs for D are: $\bigoplus_{p \lneq \infty} TA^{0,0}(X_p^{an}) \oplus \mathcal{A}^{0,0}(X(\mathbb{C})) =: \overset{0,0}{TA}(X)$

$\Rightarrow \quad 0 \longrightarrow TA^{0,0}(X) \longrightarrow \widehat{TZ}^1(X) \longrightarrow \mathrm{Div}(X) \longrightarrow 0$

Def. $\widehat{TCH}^1(X) := \widehat{TZ}^1(X) \Big/ \Big\langle \mathrm{div}(f), (-\log \|f\|_p)_p, \bigoplus_{p \leq \infty} \underbrace{\mathcal{H}(X_p)}_{\simeq \mathbb{R}^{\pi_0(X_p)}} \Big\rangle$

§3 Comparison with previous def.

Prop. $\exists$ canonical, i.e. compatible with $\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \cup\!\shortmid & & \cup\!\shortmid \\ X' & \longrightarrow & X \end{array}$:

$$\begin{array}{ccc} \hat{Z}^1(X) & \hookrightarrow & \widehat{TZ}^1(X) \\ \downarrow & & \downarrow \\ \widehat{CH}^1(X) & \longrightarrow & \widehat{TCH}^1(X) \end{array}$$

$\mathfrak{X}/\mathrm{Spec}\,\mathbb{Z}$ reg flat proj model

Compat w/ intersection products on $\widehat{TCH}^1$ and $\widehat{CH}^1$